

Generalized refracted Lévy process and its application to exit problem

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Abstract

Generalizing Kyprianou–Loeffen’s refracted Lévy processes, we define a new refracted Lévy process which is a Markov process whose positive and negative motions are Lévy processes different from each other. To construct it we utilize the excursion theory. We study its exit problem and the potential measures of the killed processes. We also discuss approximation problem.

1 Introduction

Exit problem of a real-valued stochastic process $Z = \{Z_t : t \geq 0\}$ is the problem to characterize the law of the first time of exiting an interval $[b, a]$ for $b < a$. In this paper, we are interested in the Laplace transform

$$\mathbb{E}_x^Z \left(e^{-q\tau_a^+} : \tau_a^+ < \tau_b^- \right) \quad (1.1)$$

for $q \geq 0$ and a starting point $x \in [b, a]$, where

$$\tau_a^+ = \inf\{t > 0 : Z_t > a\} \text{ and } \tau_b^- = \inf\{t > 0 : Z_t < b\}. \quad (1.2)$$

When Z is a spectrally negative Lévy process Z , it is well known that

$$\mathbb{E}_x^Z \left(e^{-q\tau_a^+} : \tau_a^+ < \tau_b^- \right) = \frac{W_Z^{(q)}(x - b)}{W_Z^{(q)}(a - b)}, \quad (1.3)$$

where $W_Z^{(q)}$ is the q -scale function of Z .

Kyprianou and Loeffen [8] have studied the exit problem when Z was a *refracted Lévy process* U , which was defined as the strong solution of the stochastic differential equation

$$U_t - U_0 = X_t - X_0 + \alpha \int_0^t 1_{\{U_s < 0\}} ds \quad t \geq 0, \quad (1.4)$$

where the driving noise X is a spectrally negative Lévy process and α is a positive constant. Define $Y_t = X_t + \alpha t$. Then we see that

$$U_t - U_s = \begin{cases} X_t - X_s & \text{whenever } U_r \geq 0 \text{ for any } r \in [s, t] \\ Y_t - Y_s & \text{whenever } U_r < 0 \text{ for any } r \in [s, t]. \end{cases} \quad (1.5)$$

They proved that the Laplace transform (1.1) for $Z = U$ takes the form

$$\mathbb{E}_x^U \left(e^{-q\tau_a^+} : \tau_a^+ < \tau_b^- \right) = \frac{W_U^{(q)}(x, b)}{W_U^{(q)}(a, b)}, \quad (1.6)$$

where the function $W_U^{(q)}$ is defined by

$$W_U^{(q)}(x, y) = W_Y^{(q)}(x - y) + \alpha 1_{(x \geq 0)} \int_0^x W_X^{(q)}(x - z) W_Y^{(q)'}(z - y) dz. \quad (1.7)$$

with $W_Y^{(q)}$ being the q -scale function of Y . They obtained, in addition, a representation of the potential measures of U using scale functions of X and Y .

A spectrally negative Lévy process can be regarded as the capital of an insurance company and applied to evaluate the risk of ruin. Hence it is sometimes called a *Lévy insurance risk process*. The Kyprianou–Loeffen’s refracted Lévy process U can be regarded as a modified insurance risk process when dividends are being paid out at a rate α during the period it exceeds 0.

In this paper, we generalize Kyprianou–Loeffen’s refracted Lévy processes. For two Lévy processes X and Y which may have different Lévy measures, we construct a new refracted process whose positive and negative motions have the same law as X and Y , respectively. More precisely,

$$\begin{cases} \text{If } x > 0, (U_t)_{t \leq \tau_0^-} & \text{under } \mathbb{P}_x^U \stackrel{\text{law}}{=} (X_t)_{t \leq \tau_0^-} & \text{under } \mathbb{P}_x^X \\ \text{If } x < 0, (U_t)_{t \leq \tau_0^+} & \text{under } \mathbb{P}_x^U \stackrel{\text{law}}{=} (Y_t)_{t \leq \tau_0^+} & \text{under } \mathbb{P}_x^Y. \end{cases} \quad (1.8)$$

One may expect that we can characterize such a process as a solution to the following stochastic differential equation

$$U_t - U_0 = \int_{(0,t]} 1_{\{U_{s-} \geq 0\}} dX_s + \int_{(0,t]} 1_{\{U_{s-} < 0\}} dY_s, \quad (1.9)$$

where the driving noises X and Y are supposed to be independent. Although (1.4) for $Y_t \stackrel{d}{=} X_t + \alpha t$ is apparently different from (1.9) because of independence, their solutions are actually equivalent in law. When X has bounded variation paths, we can construct a solution of (1.9) by a simple method of piecing excursions (see [8]); otherwise we do not know existence of a solution of (1.9). When X and Y are compound Poisson processes with positive drifts, uniqueness of the solution is easily proved because of the fact that the point 0 is irregular for itself for any solution U ; otherwise we do not know uniqueness of a solution of (1.9).

In this paper we utilize the excursion theory instead of a stochastic differential equation. Let X and Y be two spectrally negative Lévy processes. Suppose X has unbounded variation paths and has no Gaussian component. We then define the excursion measure n^U by

$$n^U \left(F \left((U_t)_{t < \tau_0^-}, (U_{t+\tau_0^-})_{t \geq 0} \right) \right) = n^X \left(\mathbb{E}_y^{Y^0} \left(F(w, (Y_t^0)_{t \geq 0}) \right) \Big|_{\substack{y=X(\tau_0^-) \\ w=(X(t))_{t < \tau_0^-}} \right), \quad (1.10)$$

where n^X stands for an excursion measure of X and $Y_t^0 = Y_{t \wedge T_0}$ for the stopped process of Y upon hitting zero. We define the stopped process $\mathbb{P}_x^{U^0}$ by replacing n^X in (1.10) by $\mathbb{P}_x^{X^0}$. We can therefore construct a Feller process from n^U together with the family of stopped processes $\left\{ \mathbb{P}_x^{U^0} \right\}_{x \neq 0}$. As one of our main theorems, we show the Laplace transform (1.1) for the process $Z = U$, our new refracted Lévy process, takes the same form as (1.6) where $W_U^{(q)}$ will be defined in (6.11) in a more complicated form than (1.7). Note that $W_U^{(q)}$'s will be represented using only Laplace exponents and scale functions of X and Y . Furthermore, we will study the potential measures of U with and without absorbing barriers.

We finally discuss approximation problem. Let X and Y as in the previous paragraph. Let $X^{(n)}$ and $Y^{(n)}$ be the compound Poisson processes with positive drifts obtained by removing small jumps of magnitude less than $\frac{1}{n}$ from X and Y , respectively. Assuming that $X^{(n)}$ and $Y^{(n)}$ are independent, we construct $U^{(n)}$ as the unique solution of (1.9). We thus show that $U^{(n)}$ converges to our refracted process U in law on the space of càdlàg paths equipped with the Skorokhod topology.

The organization of the present paper is as follows. In Section 2 we propose some notation and recall preliminary facts about spectrally negative Lévy processes. In Section 3 we calculate several quantities related to excursion measures and scale functions. In Section 4 we recall Kyprianou–Loeffen's refracted Lévy processes. In Section 5 we define our new refracted Lévy processes. In Section 6 we study the exit problem of our refracted Lévy processes. In Section 7 we calculate the potential measures of our refracted Lévy processes killed upon exiting $[b, a]$. In Section 8 we study approximation of our refracted Lévy processes by compound Poisson processes with positive drifts which are obtained by removing small jumps.

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2 Notation and preliminaries

Let \mathbb{D} denote the set of functions $\omega : [0, \infty) \rightarrow \mathbb{R}$ which are càdlàg. We equip \mathbb{D} with the Skorokhod topology. Let $\mathcal{B}(\mathbb{D})$ denote the class of Borel sets of \mathbb{D} .

When we consider a process $Z = \{Z_t : t \geq 0\} = \{Z(t) : t \geq 0\}$, we always write \mathbb{P}_x^Z for the underlying probability measure for Z starting from x . In addition to the passage times τ_a^+ and τ_b^- defined in (1.2), we sometimes need the hitting time of a point $x \in \mathbb{R}$ denoted by

$$T_x = \inf\{t > 0 : Z_t = x\}. \quad (2.1)$$

For $q > 0$, $x \in \mathbb{R}$ and a non-negative or bounded measurable function f , we write

$$R_Z^{(q)} f(x) := \mathbb{E}_x^Z \left(\int_0^\infty e^{-qt} f(Z_t) dt \right). \quad (2.2)$$

If densities exist, we write

$$R_Z^{(q)} f(x) = \int_{\mathbb{R}} r_Z^{(q)}(x, y) f(y) dy. \quad (2.3)$$

We sometimes settle a lower barrier $b < 0$ and an upper barrier $a > 0$. For $q > 0$, $x \in \mathbb{R}$ and a non-negative or bounded measurable function f , we write

$$\overline{R}_Z^{(q;b,a)} f(x) := \overline{R}_Z^{(q)} f(x) := \mathbb{E}_x^Z \left(\int_0^{\tau_a^+ \wedge \tau_b^-} e^{-qt} f(Z_t) dt \right), \quad (2.4)$$

$$\underline{R}_Z^{(q;b)} f(x) := \underline{R}_Z^{(q)} f(x) := \mathbb{E}_x^Z \left(\int_0^{\tau_b^-} e^{-qt} f(Z_t) dt \right), \quad (2.5)$$

$$\overline{R}_Z^{(q;a)} f(x) := \overline{R}_Z^{(q)} f(x) := \mathbb{E}_x^Z \left(\int_0^{\tau_a^+} e^{-qt} f(Z_t) dt \right). \quad (2.6)$$

If densities exist, we denote them by $\overline{r}_Z^{(q;b,a)}(x, y) = \overline{r}_Z^{(q)}(x, y)$ and so on, which satisfy the counterpart of (2.3).

Let Z be a spectrally negative Lévy process, which is always assumed not to be monotone. Then it is well known that the Laplace exponent

$$\Psi_Z(q) := \log \mathbb{E}_0^Z(e^{qZ_1}) \quad (2.7)$$

is finite for all $q \geq 0$. We denote its right inverse by

$$\Phi_Z(\theta) = \inf\{q \geq 0 : \Psi_Z(q) = \theta\}, \quad (2.8)$$

which is finite for all $\theta \geq 0$. If Z has bounded variation paths, the Laplace exponent is known to necessarily take the form

$$\Psi_Z(q) = \delta_Z q - \int_{(-\infty, 0)} (1 - e^{qy}) \Pi_Z(dy) \quad (2.9)$$

for some constant $\delta_Z > 0$ and some Lévy measure Π_Z satisfying $\Pi_Z[0, \infty) = 0$ and $\int_{(-\infty, 0)} (1 \wedge |y|) \Pi_Z(dy) < \infty$. If Z has unbounded variation paths, the Laplace exponent is known to necessarily take the form

$$\Psi_Z(q) = \gamma_Z q + \frac{\sigma_Z^2}{2} q^2 - \int_{(-\infty, 0)} (1 - e^{qy} + qy 1_{(-1, 0)}(y)) \Pi_Z(dy) \quad (2.10)$$

for some constants $\gamma_Z \in \mathbb{R}$ and $\sigma_Z \geq 0$ and some Lévy measure Π_Z satisfying $\Pi_Z[0, \infty) = 0$ and $\int_{(-\infty, 0)} (1 \wedge y^2) \Pi_Z(dy) < \infty$.

Definition 2.1. For each $q \geq 0$, we define $W_Z^{(q)} : \mathbb{R} \rightarrow [0, \infty)$ such that $W_Z^{(q)} = 0$ on $(-\infty, 0)$ and $W_Z^{(q)}$ on $[0, \infty)$ is continuous satisfying

$$\int_0^\infty e^{-\beta x} W_Z^{(q)}(x) dx = \frac{1}{\Psi_Z(\beta) - q} \quad (2.11)$$

for all $\beta > \Phi_Z(q)$. This function $W_Z^{(q)}$ is called the q -scale function of Z .

For the proof and its basic facts listed below, see, e.g., [7]. For all $b < x < a$ and $q \geq 0$, we have

$$\mathbb{E}_x^Z \left(e^{-q\tau_a^+} : \tau_a^+ < \tau_b^- \right) = \frac{W_Z^{(q)}(x - b)}{W_Z^{(q)}(a - b)} \quad (2.12)$$

and

$$\mathbb{E}_x^Z \left(e^{-q\tau_a^+} : \tau_a^+ < \infty \right) = e^{-\Phi_Z(q)(a-x)}. \quad (2.13)$$

It is known that, when Z has bounded variation paths, we have

$$W_Z^{(q)}(0) = \frac{1}{\delta_Z} \quad (2.14)$$

for all $q \geq 0$. For all $y \in \mathbb{R}$ and $q \geq 0$, we have

$$r_Z^{(q)}(x, y) = \Phi_Z'(q) e^{-\Phi_Z(q)(y-x)} - W_Z^{(q)}(x - y), \quad x \in \mathbb{R}, \quad (2.15)$$

$$\underline{r}_Z^{(q;b)}(x, y) = \underline{r}_Z^{(q)}(x, y) = e^{-\Phi_Z(q)(y-b)} W_Z^{(q)}(x - b) - W_Z^{(q)}(x - y), \quad x \in [b, \infty), \quad (2.16)$$

$$\bar{r}_Z^{(q;a)}(x, y) = \bar{r}_Z^{(q)}(x, y) = e^{-\Phi_Z(q)(a-x)} W_Z^{(q)}(a - y) - W_Z^{(q)}(x - y), \quad x \in (-\infty, a] \quad (2.17)$$

and

$$\bar{\underline{r}}_Z^{(q;b,a)}(x, y) = \bar{\underline{r}}_Z^{(q)}(x, y) = \frac{W_Z^{(q)}(x - b) W_Z^{(q)}(a - y)}{W_Z^{(q)}(a - b)} - W_Z^{(q)}(x - y), \quad x \in [b, a]. \quad (2.18)$$

We write $\tilde{\Pi}_z$ for the measure carried on $(-\infty, 0) \times (0, \infty)$ defined by

$$\tilde{\Pi}_Z(du \, dv) := \Pi_Z(du - v)dv. \quad (2.19)$$

Theorem 2.2 (see, e.g., [7, Theorem 10.1]). *For all $0 < x < \infty$, $q \geq 0$, and non-negative measurable function $f : \mathbb{R}^2 \rightarrow [0, \infty)$, we have*

$$\mathbb{E}_x^Z \left(e^{-q\tau_0^-} f(Z_{\tau_0^-}, Z_{\tau_0^- -}) : \tau_0^- < \infty, Z_{\tau_0^-} < 0 \right) = \int f(u, v) G_Z^{(q)}(x, du \, dv), \quad (2.20)$$

where $G_Z^{(q)}(x, \cdot)$ is a measure carried on $(-\infty, 0) \times (0, \infty)$ defined by

$$G_Z^{(q)}(x, du \, dv) := \underline{r}_Z^{(q;0)}(x, v) \tilde{\Pi}_Z(du \, dv). \quad (2.21)$$

This kernel $G_Z^{(q)}$ is called the *Gerber–Shiu measure*. The following is a slight refinement of Theorem 2.2.

Lemma 2.3. *For all $0 < x < a$, $q \geq 0$, and non-negative measurable function f , we have*

$$\mathbb{E}_x^Z \left(e^{-q\tau_0^-} f(Z_{\tau_0^-}, Z_{\tau_0^- -}) : \tau_0^- < \tau_a^+, Z_{\tau_0^-} < 0 \right) = \int f(u, v) \overline{G}_Z^{(q,a)}(x, du \, dv), \quad (2.22)$$

where $\overline{G}_Z^{(q,a)}(x, \cdot)$ is a measure carried on $(-\infty, 0) \times (0, \infty)$ defined by

$$\overline{G}_Z^{(q,a)}(x, du \, dv) = \overline{G}_Z^{(q)}(x, du \, dv) := \underline{\overline{G}}_Z^{(q;0,a)}(x, v) \tilde{\Pi}_Z(du \, dv). \quad (2.23)$$

Proof. By the strong Markov property, we have

$$\begin{aligned} & \mathbb{E}_x^Z \left(e^{-q\tau_0^-} f(Z_{\tau_0^-}, Z_{\tau_0^- -}) : \tau_a^+ < \tau_0^- < \infty, Z_{\tau_0^-} < 0 \right) \\ &= \mathbb{E}_x^Z \left(e^{-q\tau_a^+} \left(e^{-q\tau_0^-} f(Z_{\tau_0^-}, Z_{\tau_0^- -}) 1_{(\tau_0^- < \infty, Z_{\tau_0^-} < 0)} \right) \circ \theta_{\tau_a^+} : \tau_a^+ < \tau_0^- < \infty \right) \end{aligned} \quad (2.24)$$

$$= \frac{W_Z^{(q)}(x)}{W_Z^{(q)}(a)} \int f(u, v) G_Z^{(q)}(a, du \, dv). \quad (2.25)$$

Since $\{\tau_0^- < \tau_a^+\} = \{\tau_0^- < \infty\} - \{\tau_a^+ < \tau_0^- < \infty\}$, we obtain (2.22). \square

3 Some calculations related to excursion measures and scale functions

In this section, we make some calculations related to excursion measures and scale functions for a spectrally negative Lévy process X . See [9], [1] and [10] for a close relation between n^X , the excursion measure of X itself, and the excursion measure of the reflected of X . We divide the discussion into the two cases of unbounded and bounded variations.

(I) We assume that

$$X \text{ has unbounded variation paths and have no Gaussian component.} \quad (3.1)$$

Since 0 is regular for itself, X has an excursion measure n^X away from zero. We impose on n^X the following normalization:

$$n^X(1 - e^{-qT_0}) = \frac{1}{r_X^{(q)}(0, 0)} = \frac{1}{\Phi_X'(q)} = \Psi_X'(\Phi_X(q)). \quad (3.2)$$

Note that n^X is carried on the set of càdlàg paths stopped upon hitting 0. Note also that n^X possesses the Markov property; for example,

$$n^X(X_s \in B_1, X_t \in B_2) = n^X \left(1_{\{X_s \in B_1\}} \mathbb{P}_{X(s)}^{X_0}(X_{t-s}^0 \in B_2) \right), \quad (3.3)$$

for all $0 < s < t$ and $B_1, B_2 \in \mathcal{B}(\mathbb{R})$ where $X_t^0 = X_{t \wedge T_0}$ denote the stopped process of X upon hitting zero. Since X has no Gaussian component, we can see

$$0 < \tau_0^- < T_0 \leq \infty \quad \text{or} \quad \tau_0^- = T_0 = \infty \quad n^X\text{-a.e.} \quad (3.4)$$

by [9, Theorem 3].

Theorem 3.1. *For all $a > 0$ and $q \geq 0$, we have*

$$n^X \left(e^{-q\tau_a^+} : \tau_a^+ < \infty \right) = \frac{1}{W_X^{(q)}(a)}. \quad (3.5)$$

In particular, we have

$$n^X (\tau_a^+ < \infty) = \frac{1}{W_X(a)}, \quad (3.6)$$

where $W_X := W_X^{(0)}$.

Remark 3.2. The two ways of normalization (3.2) and (3.6) are natural analogies of those for diffusion processes. See [3, (2.5) and Theorem 3.1] and [13, (39) and Theorem 3.1].

The following theorem can be regarded as the Gerber–Shiu measure for the excursion measure (See also [10]).

Theorem 3.3. *For all $q \geq 0$ and non-negative measurable function f , we have*

$$n^X \left(e^{-q\tau_0^-} f(X_{\tau_0^-}, X_{\tau_0^- -}) : \tau_0^- < \infty \right) = \int f(u, v) K_X^{(q)}(du \, dv), \quad (3.7)$$

where $K_X^{(q)}$ is a measure carried on $(-\infty, 0) \times (0, \infty)$ defined by

$$K_X^{(q)}(du \, dv) = e^{-\Phi_X(q)v} \tilde{\Pi}_X(du \, dv). \quad (3.8)$$

We prove Theorems 3.1 and 3.3 at the same time.

Proof of Theorems 3.1 and 3.3.

Step.1 We show that the quantity

$$c := n^X \left(e^{-q\tau_a^+} : \tau_a^+ < \infty \right) W_X^{(q)}(a) \quad (3.9)$$

does not depend upon $a > 0$ nor $q \geq 0$. First, we prove

$$n^X (\tau_a^+ < \infty) W_X(a) = n^X \left(e^{-q\tau_a^+} : \tau_a^+ < \infty \right) W_X^{(q)}(a) \quad (3.10)$$

for all $a > 0$ and $q \geq 0$. Using the monotone convergence theorem, we have

$$\frac{n^X(e^{-q\tau_a^+} : \tau_a^+ < \infty)}{n^X(\tau_a^+ < \infty)} = \lim_{\varepsilon \downarrow 0} \frac{n^X(e^{-q\tau_\varepsilon^+} (e^{-q\tau_a^+} 1_{\{\tau_a^+ < \infty\}}) \circ \theta_{\tau_\varepsilon^+} : \tau_\varepsilon^+ < \infty)}{n^X(e^{-q\tau_\varepsilon^+} (1_{\{\tau_a^+ < \infty\}}) \circ \theta_{\tau_\varepsilon^+} : \tau_\varepsilon^+ < \infty)}. \quad (3.11)$$

Using the strong Markov property and (2.12), we have

$$(3.11) = \lim_{\varepsilon \downarrow 0} \frac{n^X(e^{-q\tau_\varepsilon^+} : \tau_\varepsilon^+ < \infty) \mathbb{E}_\varepsilon^X(e^{-q\tau_a^+} 1_{\{\tau_a^+ < \tau_0^-\}})}{n^X(e^{-q\tau_\varepsilon^+} : \tau_\varepsilon^+ < \infty) \mathbb{P}_\varepsilon^X(\tau_a^+ < \tau_0^-)} = \frac{W_X(a)}{W_X^{(q)}(a)}, \quad (3.12)$$

where we used $\lim_{\varepsilon \downarrow 0} W_X^{(q)}(\varepsilon)/W_X(\varepsilon) = 1$ from [4, Lemma 1. (i)]. Second, we prove

$$n^X(\tau_{a_1}^+ < \infty) W_X(a_1) = n^X(\tau_{a_2}^+ < \infty) W_X(a_2), \quad (3.13)$$

for all $0 < a_1 < a_2$. This identity can be obtained by

$$\frac{n^X(\tau_{a_2}^+ < \infty)}{n^X(\tau_{a_1}^+ < \infty)} = \frac{n^X((\tau_{a_2}^+ < \tau_0^-) \circ \theta_{\tau_{a_1}^+} : \tau_{a_1}^+ < \infty)}{n^X(\tau_{a_1}^+ < \infty)} = \frac{W_X(a_1)}{W_X(a_2)}, \quad (3.14)$$

where we used the strong Markov property and (2.12).

Step.2 We show (3.7) with $K_X^{(q)}$ being multiplied by c . Using the monotone convergence theorem and the strong Markov property, we have

$$\begin{aligned} & n^X(e^{-q\tau_0^-} f(X_{\tau_0^-}, X_{\tau_0^- -}) : \tau_0^- < \infty) \\ &= \lim_{\varepsilon \downarrow 0} n^X \left(e^{-q\tau_\varepsilon^+} \left(e^{-q\tau_0^-} f(X_{\tau_0^-}, X_{\tau_0^- -}) 1_{\{X_{\tau_0^- -} > \varepsilon\}} \right) \circ \theta_{\tau_\varepsilon^+} : \tau_\varepsilon^+ < \infty, \tau_0^- < \infty \right) \end{aligned} \quad (3.15)$$

and using Theorem 2.2 and **Step.1**, we have

$$(3.15) = \lim_{\varepsilon \downarrow 0} \frac{c}{W_X^{(q)}(\varepsilon)} W_X^{(q)}(\varepsilon) \int_\varepsilon^\infty dv \int_{(-\infty, 0)} e^{-\Phi_X(q)v} f(u, v) \Pi_X(du - v) \quad (3.16)$$

$$= \int f(u, v) c K_X^{(q)}(du, dv). \quad (3.17)$$

Step.3 We show $c = 1$. Since X has no Gaussian component, differentiating (2.10) with $\sigma_X = 0$, we have

$$\Psi'_X(q) = q_X + \int_{(-\infty, 0)} (ye^{qy} - y1_{(-1, 0)}(y)) \Pi_X(dy) \quad (3.18)$$

for all $q > 0$. Using (3.2), we have on one hand

$$n^X(1 - e^{-qT_0}) = \Psi'_X(\Phi_X(q)) = \gamma_X + \int_{(-\infty, 0)} (ye^{\Phi_X(q)y} - y1_{(-1, 0)}(y)) \Pi_X(dy). \quad (3.19)$$

On the other hand, using the monotone convergence theorem and the strong Markov property, we have

$$\begin{aligned} & n^X(1 - e^{-qT_0}) \\ &= n^X(\tau_0^- = \infty) + n^X(1 - e^{-qT_0} : \tau_0^- < \infty) \end{aligned} \quad (3.20)$$

$$= n^X(\tau_1^+ < \infty) \lim_{p \uparrow \infty} \mathbb{E}_1^X(\tau_p^+ < \tau_0^-) + n^X\left(1 - e^{-q\tau_0^-} \mathbb{E}_{X(\tau_0^-)}^X(e^{-qT_0}) : \tau_0^- < \infty\right). \quad (3.21)$$

Using (2.12), (2.13), **Step.1** and **Step.2**, we have

$$(3.21) = n^X(\tau_1^+ < \infty) \lim_{p \uparrow \infty} \frac{W_X(1)}{W_X(p)} + n^X\left(1 - e^{-q\tau_0^- + \Phi_X(q)X(\tau_0^-)} : \tau_0^- < \infty\right) \quad (3.22)$$

$$= c \frac{1}{W_X(\infty)} + c \int (e^{-\Phi_X(0)v} - e^{\Phi_X(q)(u-v)}) \tilde{\Pi}_X(du \, dv). \quad (3.23)$$

Since it is known that

$$W_X(\infty) = \begin{cases} \frac{1}{\Psi_X'(0+)} & \mathbb{P}(\lim_{t \uparrow \infty} X = \infty) = 1 \\ \infty & \text{otherwise} \end{cases} \quad (3.24)$$

(see e.g., [7, pp.247]), we have

$$(3.23) = c(\Psi_X'(0+) \vee 0) + c \int_{(-\infty, 0)} \Pi_X(du) \int_0^{-u} (e^{-\Phi_X(0)v} - e^{\Phi_X(q)u}) \, dv. \quad (3.25)$$

We divide the remainder of the proof into two parts.

(i) Suppose $\Psi_X'(0+) > 0$. In this case, we have $\Phi_X(0) = 0$ and so

$$(3.25) = c\Psi_X'(0+) + c \int_{(-\infty, 0)} (ue^{\Phi_X(q)u} - u) \Pi_X(du). \quad (3.26)$$

Using (3.18), we have

$$(3.26) = c\left(\gamma_X + \int_{(-\infty, 0)} (ue^{\Phi_X(q)u} - u1_{(-1, 0)}(u)) \Pi_X(du)\right). \quad (3.27)$$

Using (3.19), we obtain $c = 1$.

(ii) Suppose $\Psi_X'(0+) \leq 0$. In this case, we have

$$(3.25) = c \int_{(-\infty, 0)} \left(ue^{\Phi_X(q)u} + \frac{1}{\Phi_X(0)} - \frac{1}{\Phi_X(0)}e^{\Phi_X(0)u}\right) \Pi_X(du). \quad (3.28)$$

Since $\Psi(\Phi(0)) = 0$ and by (2.10) with $\sigma_X = 0$, we have

$$(3.28) = c\left(\gamma_X + \int_{(-\infty, 0)} (ue^{\Phi_X(q)u} - u1_{(-1, 0)}(u)) \Pi_X(du)\right). \quad (3.29)$$

Using (3.19), we obtain $c = 1$. Thus the proof is complete. \square

We prove two lemmas for later use.

Lemma 3.4. *For all $a > 0$, $q \geq 0$ and non-negative measurable function f we have*

$$n^X \left(e^{-q\tau_0^-} f \left(X_{\tau_0^-}, X_{\tau_0^- -} \right) : \tau_0^- < \tau_a^+ \right) = \int f(u, v) \overline{K}_X^{(q, a)}(du \, dv), \quad (3.30)$$

where $\overline{K}_X^{(q, a)}$ is a measure carried on $(-\infty, 0) \times (0, \infty)$ defined by

$$\overline{K}_X^{(q, a)}(du \, dv) = \overline{K}_X^{(q)}(du \, dv) := \frac{W_X^{(q)}(a - v)}{W_X^{(q)}(a)} \tilde{\Pi}_X(du \, dv) \quad (3.31)$$

The proof is parallel to that of Lemma 2.2, so that we omit it.

Lemma 3.5. *For all $q \geq 0$ and non-negative measurable function f , we have*

$$n^X \left(\int_0^{\tau_0^- \wedge T_0} e^{-qt} f(X_t) dt \right) = \int_0^\infty e^{-\Phi_X(q)y} f(y) dy. \quad (3.32)$$

Proof. Using the monotone convergence theorem, we have

$$n^X \left(\int_0^{\tau_0^- \wedge T_0} e^{-qt} f(X_t) dt \right) = \lim_{\varepsilon \downarrow 0} n^X \left(\int_{\tau_\varepsilon^+}^{\tau_0^- \wedge T_0} e^{-qt} f(X_t) dt : \tau_\varepsilon^+ < \infty \right) \quad (3.33)$$

and using the strong Markov property, we have

$$(3.33) = \lim_{\varepsilon \downarrow 0} n^X \left(e^{-q\tau_\varepsilon^+} \left(\int_0^{\tau_0^- \wedge T_0} e^{-qt} f(X_t) dt \right) \circ \theta_{\tau_\varepsilon^+} : \tau_\varepsilon^+ < \infty \right) \quad (3.34)$$

$$= \lim_{\varepsilon \downarrow 0} n^X \left(e^{-q\tau_\varepsilon^+} : \tau_\varepsilon^+ < \infty \right) \mathbb{E}_\varepsilon^X \left(\int_0^{\tau_0^- \wedge T_0} e^{-qt} f(X_t) dt \right) \quad (3.35)$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{W_X^{(q)}(\varepsilon)} \mathbb{E}_\varepsilon^X \left(\int_0^{\tau_0^- \wedge T_0} e^{-qt} f(X_t) dt \right), \quad (3.36)$$

where in (3.36) we used Theorem 3.1. Using (2.16) with $b = 0$, we obtain

$$\int_\varepsilon^\infty f(y) e^{-\Phi_X(q)y} dy \leq \frac{1}{W_X^{(q)}(\varepsilon)} \mathbb{E}_\varepsilon^X \left(\int_0^{\tau_0^-} e^{-qt} f(X_t) dt \right) \leq \int_0^\infty f(y) e^{-\Phi_X(q)y} dy. \quad (3.37)$$

By the monotone convergence theorem, the proof is completed. \square

(II) We assume that X has bounded variation paths. Note that in this case 0 is irregular for itself. We write

$$n^X = \delta_X \mathbb{P}_0^X. \quad (3.38)$$

Then we have

$$n^X\left(e^{-q\tau_a^+} : \tau_a^+ < \infty\right) = \delta_X \mathbb{E}_0^X\left(e^{-q\tau_a^+} : \tau_a^+ < \infty\right) = \delta_X \frac{W_X^{(q)}(0)}{W_X^{(q)}(a)} = \frac{1}{W_X^{(q)}(a)}, \quad (3.39)$$

where we used (2.12) and (2.14). Thus we see that Theorem 3.1 still holds in this case. Lemmas 3.3, 3.4 (with $c = 1$) and 3.5 still hold as they are by a similar argument. In particular, we obtain

$$n^X(1 - e^{-qT_0}) = \delta_X \mathbb{E}_0^X(1 - e^{-qT_0}) = \frac{1}{\Phi_X'(q)} = \Psi'(\Phi_X(q)). \quad (3.40)$$

which may be regarded as the counterpart of the normalization (3.2) in the unbounded case.

4 Kyprianou–Loeffen’s Refracted Lévy processes

We fix a constant $\alpha > 0$, and let X be a general spectrally negative Lévy process. Kyprianou–Loeffen’s refracted Lévy process introduced the following stochastic differential equation driven by X :

$$U_t - U_0 = X_t - X_0 + \alpha \int_0^t 1_{\{U_s < 0\}} ds. \quad (4.1)$$

If the process U is a solution, U behaves as X when U exceeds 0 and as $Y_t := X_t + \alpha t$ when U does not exceed 0.

Note that $0 < \delta_X < \delta_X + \alpha = \delta_Y$ if X has bounded variation paths. Let us recall results of Kyprianou–Loeffen [8]. Note that for their results X may possibly have Gaussian component.

Theorem 4.1 ([8]). *For a fixed starting point $U_0 = x \in \mathbb{R}$, there exists a unique strong solution to (4.1).*

Theorem 4.2 ([8]). *For all $x \in [b, a]$ and $q \geq 0$, we have (1.6) where $W_U^{(q)}$ is defined by (1.7).*

They also calculated the potential densities with and without barriers.

Theorem 4.3 ([8]). *For all $x \in [b, a]$, $q > 0$, we have*

$$\bar{r}_U^{(q)}(x, y) = \begin{cases} \frac{W_U^{(q)}(x, b)}{W_U^{(q)}(a, b)} W_X^{(q)}(a - y) - W_X^{(q)}(x - y) & y \in (0, a] \\ \frac{W_U^{(q)}(x, b)}{W_U^{(q)}(a, b)} W_U^{(q)}(a, y) - W_U^{(q)}(x, y) & y \in [b, 0], \end{cases} \quad (4.2)$$

$$\underline{r}_U^{(q)}(x, y) = \begin{cases} \frac{W_U^{(q)}(x, b)}{\alpha \int_0^\infty e^{-\Phi_X(q)(z-b)} W_Y^{(q)'}(z-b) dz} e^{-\Phi_X(q)(y-b)} - W_X^{(q)}(x - y) & y \in (0, \infty) \\ \frac{\int_0^\infty e^{-\Phi_X(q)(z-b)} W_Y^{(q)'}(z-y) dz}{\int_0^\infty e^{-\Phi_X(q)(z-b)} W_Y^{(q)'}(z-b) dz} W_U^{(q)}(x, b) - W_U^{(q)}(x, y) & y \in [b, 0] \end{cases} \quad (4.3)$$

$$\bar{r}_U^{(q)}(x, y) = \begin{cases} \frac{e^{\Phi_Y(q)(x-b)} + \alpha \Phi_Y(q) \int_0^x e^{\Phi_Y(q)(z-b)} W_X^{(q)}(x-z) dz}{e^{\Phi_Y(q)(a-b)} + \alpha \Phi_Y(q) \int_0^a e^{\Phi_Y(q)(z-b)} W_X^{(q)}(a-z) dz} W_X^{(q)}(a - y) - W_X^{(q)}(x - y) & y \in (0, a] \\ \frac{e^{\Phi_Y(q)(x-b)} + \alpha \Phi_Y(q) \int_0^x e^{\Phi_Y(q)(z-b)} W_X^{(q)}(x-z) dz}{e^{\Phi_Y(q)(a-b)} + \alpha \Phi_Y(q) \int_0^a e^{\Phi_Y(q)(z-b)} W_X^{(q)}(a-z) dz} W_U^{(q)}(a, y) - W_U^{(q)}(x, y) & y \in [b, 0] \end{cases} \quad (4.4)$$

$$r_U^{(q)}(x, y) = \begin{cases} \left(e^{\Phi_Y(q)x} + \alpha \Phi_Y(q) e^{\Phi_Y(q)b} \int_0^x e^{\Phi_Y(q)(z-b)} W_X^{(q)}(x-z) dz \right) \times \frac{\Phi_X(q) - \Phi_Y(q)}{\Phi_Y(q)} e^{-\Phi_X(q)y} - W_X^{(q)}(x - y) & y \in (0, a] \\ \left(e^{\Phi_Y(q)x} + \alpha \Phi_Y(q) e^{\Phi_Y(q)b} \int_0^x e^{\Phi_Y(q)(z-b)} W_X^{(q)}(x-z) dz \right) \frac{\Phi_X(q) - \Phi_Y(q)}{\Phi_Y(q)} e^{-\Phi_X(q)y} \times \int_0^\infty e^{-\Phi_Y(q)(z-b)} W_Y^{(q)'}(z-y) dz - W_U^{(q)}(x, y) & y \in [b, 0] \end{cases} \quad (4.5)$$

where $W_U^{(q)}$ has been given in (1.7).

5 Generalization of refracted Lévy processes

We now generalized Kyprianou–Loeffen's refracted Lévy processes. We assume that X and Y are spectrally negative Lévy processes. We assume, in addition, that

X has no Gaussian component whenever X has unbounded variation paths. (5.1)

In the unbounded variation case, we define the law of the stopped process $\mathbb{P}_x^{U^0}$ by

$$\mathbb{P}_x^{U^0} \left(F \left((U_t)_{t < \tau_0^-}, (U_{t+\tau_0^-})_{t \geq 0} \right) \right) = \mathbb{P}_x^X \left(\mathbb{E}_y^{Y^0} (F(w, (Y_t^0)_{t \geq 0})) \Big|_{\substack{y=X(\tau_0^-) \\ w=(X(t))_{t < \tau_0^-}} \right) \quad x \neq 0 \quad (5.2)$$

and the excursion measure n^U by

$$n^U \left(F \left((U_t)_{t < \tau_0^-}, (U_{t+\tau_0^-})_{t \geq 0} \right) \right) = n^X \left(\mathbb{E}_y^{Y^0} (F(w, (Y_t^0)_{t \geq 0})) \Big|_{\substack{y=X(\tau_0^-) \\ w=(X(t))_{t < \tau_0^-}} \right) \quad (1.10)$$

for all non-negative measurable functional F , where $Y_t^0 = Y_{t \wedge T_0}$ denotes the stopped process of Y upon hitting zero. Thus, using the excursion theory, we can construct the

strong Markov process U without stagnancy at 0 (that is, $R_U^{(1)}1_{\{0\}} = 0$) from n^U together with $\{\mathbb{P}_x^{U^0}\}_{x \neq 0}$.

In the bounded variation case, we define U as a solution of (1.4) constructed connecting X and Y mutually (this argument is similar as [8]). When X and Y are compound Poisson processes, uniqueness of the solution of (1.4) is easily proved. We write

$$n^X = \delta_X \mathbb{P}_0^{X^0} \text{ and } n^U = \delta_X \mathbb{P}_0^{U^0}. \quad (5.3)$$

Then we obtain (1.10) as a formula. Therefore we can do a simultaneous discussion in both of the two cases between of bounded and unbounded variation.

Theorem 5.1. *For all $q > 0$ and non-negative measurable function f with $f(0) = 0$, we have*

$$N_U^{(q)} f := n^U \left(\int_0^{T_0} e^{-qt} f(X_t) dt \right) \quad (5.4)$$

$$= \int_0^\infty e^{-\Phi_X(q)y} f(y) dy + \int R_{Y^0}^{(q)} f(u) K_X^{(q)}(du \, dv). \quad (5.5)$$

Consequently we have

$$R_U^{(q)} f(0) = \frac{N_U^{(q)} f}{q N_U^{(q)} 1}, \quad (5.6)$$

$$R_U^{(q)} f(x) = R_{Y^0}^{(q)} f(x) + e^{\Phi_Y(q)x} R_U^{(q)} f(0), \quad x < 0, \quad (5.7)$$

and

$$R_U^{(q)} f(x) = \underline{R}_X^{(q;0)} f(x) + \int R_U^{(q)} f(u) G_X^{(q)}(x, du \, dv), \quad x > 0, \quad (5.8)$$

where

$$\underline{R}_X^{(q;0)} f(x) = \mathbb{E}_x^X \left(\int_0^{\tau_0^-} e^{-qt} f(X_t) dt \right). \quad (5.9)$$

Proof. Let us calculate $N_U^{(q)} f$. Since $\int_0^{T_0} = \int_0^{\tau_0^-} + \int_{\tau_0^-}^{T_0}$, we have that $N_U^{(q)} f$ is equal to

$$n^U \left(\int_0^{\tau_0^-} e^{-qt} f(U_t) dt \right) + n^U \left(e^{-q\tau_0^-} \left(\int_0^{T_0} e^{-qt} f(U_t) dt \right) \circ \theta_{\tau_0^-} : \tau_0^- < \infty \right). \quad (5.10)$$

Using Lemmas 3.5 and 3.3, we obtain (5.5).

Let us prove (5.6). When X has unbounded variation paths, the formula (5.6) can be found, e.g., in [12, pp.423]. Suppose X has bounded variation paths. We denote $T_0^{(0)} = 0$ and define

$$T_0^{(n)} = \inf \left\{ t > T_0^{(n-1)} : X_t = 0 \right\}$$

recursively for all $n \in \mathbb{N}$. Then we have

$$R_U^{(q)} f(0) = \sum_{n=0}^{\infty} \mathbb{E}_0^U \left(\int_{T_0^{(n)}}^{T_0^{(n+1)}} e^{-qt} f(U_t) dt : T_0^{(n)} < \infty \right) \quad (5.11)$$

$$= \sum_{n=0}^{\infty} \mathbb{E}_0^U (e^{-qT_0})^n \mathbb{E}_0^U \left(\int_0^{T_0} e^{-qt} f(U_t) dt \right) \quad (5.12)$$

$$= \frac{\mathbb{E}_0^U \left(\int_0^{T_0} e^{-qt} f(U_t) dt \right)}{q \mathbb{E}_0^U \left(\int_0^{T_0} e^{-qt} dt \right)}. \quad (5.13)$$

Since we write $n^U = \delta_X \mathbb{P}_0^U$, we obtain (5.6).

The remainder of the proof is straightforward. \square

The following theorem shows the choice of n^U leads to a normalization similar to (3.2).

Theorem 5.2. *For all $q > 0$, we have*

$$n^U (1 - e^{-qT_0}) = \frac{1}{r_U^{(q)}(0, 0+)} \quad (5.14)$$

$$= \left(\Psi'_X(0) \vee 0 \right) + \int (e^{\Phi_Y(0)u} - e^{\Phi_Y(q)u - \Phi_X(q)v}) \tilde{\Pi}_X(du \, dv). \quad (5.15)$$

Proof. By (5.6) of Theorem 5.1, we have

$$r_U^{(q)}(0, y) = \frac{1}{qN_U^{(q)}1} \left(e^{-\Phi_X(q)y} 1_{(y>0)} + \int r_{Y^0}^{(q)}(u, y) K_X^{(q)}(du \, dv) \right). \quad (5.16)$$

Since $r_{Y^0}^{(q)}(u, y) = 0$ for $U < 0$ and $y > 0$, we have

$$r_U^{(q)}(0, 0+) = \frac{1}{qN_U^{(q)}1}. \quad (5.17)$$

On the other hand, we have

$$qN_U^{(q)}1 = n^U (1 - e^{-qT_0}) \quad (5.18)$$

by the definition of $N_U^{(q)}$. Thus we obtain (5.14).

The other expectation (5.15) is proved easy by a similar argument to (3.25). \square

6 Exit Problem of generalized refracted Lévy processes

We prepare a general formula.

Lemma 6.1. *Let Z be a standard process with no positive jumps without stagnancy at 0 (i.e., $R_Z^{(1)}1_{\{0\}} = 0$). If 0 is regular for itself, then*

$$\mathbb{E}_0^Z \left(e^{-q\tau_a^+} : \tau_a^+ < \tau_b^- \right) = \frac{n^Z \left(e^{-q\tau_a^+} : \tau_a^+ < \infty \right)}{n^Z \left(1 - e^{-qT_0} 1_{\{\tau_a^+ = \infty, \tau_b^- = \infty\}} \right)} \quad (6.1)$$

for all $a > 0 > b$ and $q \geq 0$, where n^Z denotes an excursion measure away from 0. In the case 0 is irregular for itself, the identity (6.1) still holds if n^Z denotes a constant multiple of \mathbb{P}_0^Z .

Proof. We assume first that 0 is regular for itself. Let p denote a Poisson point process with characteristic measure n^Z . Set $\eta(s) = \sum_{u \leq s} T_0(p(u))$. For $E \in \mathcal{B}(\mathbb{D})$, we write $\kappa_E = \inf\{s \geq 0 : p(s) \in E\}$. We let $A = \{\tau_a^+ < \infty\}$ and $B = \{\tau_a^+ = \infty, \tau_b^- < \infty\}$, and we denote by $\varepsilon^* = p(\kappa_{A \cup B})$ the first excursion belonging to $A \cup B = \{\tau_a^+ < \infty\} \cup \{\tau_b^- < \infty\}$. Then we have

$$\mathbb{E}_0^Z \left(e^{-q\tau_a^+} : \tau_a^+ < \tau_b^- \right) = \mathbb{E} \left(e^{-q\eta(\kappa_{A \cup B}-)} e^{-q\tau_a^+(\varepsilon^*)} \right) \quad (6.2)$$

$$= \mathbb{E} \left(e^{-q\eta(\kappa_{A \cup B}-)} \right) \frac{n^Z \left(e^{-q\tau_a^+} : A \cup B \right)}{n^Z(A \cup B)}, \quad (6.3)$$

where we used the renewal property of the Poisson point process. We write p' for p restricted to excursions belonging to $(A \cup B)^c$ and write $\eta'(s) = \sum_{u \leq s} T_0(p'(u))$. Since $\eta(\kappa_{A \cup B}-) = \eta'(\kappa_{A \cup B})$ where η' and $\kappa_{A \cup B}$ are independent, we have

$$\mathbb{E} \left(e^{-q\eta(\kappa_{A \cup B}-)} \right) = n^Z(A \cup B) \int_0^\infty e^{-n^Z(A \cup B)t} \mathbb{E}_0^Z \left(e^{-q\eta'(t)} \right) dt \quad (6.4)$$

$$= n^Z(A \cup B) \int_0^\infty e^{-n^Z(A \cup B)t} \left(\exp(-tn^Z(1 - e^{-qT_0} : (A \cup B)^c)) \right) dt \quad (6.5)$$

$$= \frac{n^Z(A \cup B)}{n^Z(1 - e^{-qT_0} 1_{(A \cup B)^c})}. \quad (6.6)$$

Thus we obtain (6.1).

We second assume that 0 is irregular for itself. Using the notation of the proof of Theorem 5.1, we have

$$\mathbb{E}_0^Z \left(e^{-q\tau_a^+} : \tau_a^+ < \tau_b^- \right) = \sum_{n=0}^\infty \mathbb{E}_0^Z \left(e^{-q\tau_a^+} : T_0^{(n)} < \tau_a^+ < T_0^{(n+1)} \leq \tau_b^- \right) \quad (6.7)$$

$$= \sum_{n=0}^\infty \mathbb{E}_0^Z \left(e^{-qT_0} : \tau_a^+ = \infty, \tau_b^- = \infty \right)^n \mathbb{E}_0^Z \left(e^{-q\tau_a^+} : \tau_a^+ < \infty \right) \quad (6.8)$$

$$= \frac{\mathbb{E}_0^Z \left(e^{-q\tau_a^+} : \tau_a^+ < \infty \right)}{1 - \mathbb{E}_0^Z \left(e^{-qT_0} : \tau_a^+ = \infty, \tau_b^- = \infty \right)}. \quad (6.9)$$

Thus we obtain (6.1). \square

Theorem 6.2. For all $x \in [b, a]$ and $q \geq 0$, we have

$$\mathbb{E}_x^U \left(e^{-q\tau_a^+} : \tau_a^+ < \tau_b^- \right) = \frac{W_U^{(q)}(x, b)}{W_U^{(q)}(a, b)}, \quad (6.10)$$

where

$$W_U^{(q)}(x, y) = \begin{cases} W_X^{(q)}(x)W_Y^{(q)}(-y)(\Psi_X'(0) \vee 0) \\ \quad + \int (W_X^{(q)}(x)W_Y^{(q)}(-y)e^{\Phi_Y(0)u} \\ \quad - W_Y^{(q)}(u-y)W_X^{(q)}(x-v))\tilde{\Pi}_X(du \, dv), & x \in (0, \infty) \\ W_Y^{(q)}(x-y), & x \in (-\infty, 0]. \end{cases} \quad (6.11)$$

Proof. When X has bounded variation paths, we write $n^U = \delta_X \mathbb{E}_0^U$. Then Lemma 3.3, Lemma 3.4 and 3.5 still hold true even when X has bounded variation paths.

Let us consider the case of unbounded variation. First we calculate $\mathbb{E}_0^U \left(e^{-q\tau_a^+} : \tau_a^+ < \tau_b^- \right)$. Using Lemma 6.1, we have

$$\mathbb{E}_0^U \left(e^{-q\tau_a^+} : \tau_a^+ < \tau_b^- \right) = \frac{n^U \left(e^{-q\tau_a^+} : \tau_a^+ < \infty \right)}{n^U \left(1 - e^{-qT_0} 1_{\{\tau_a^+ = \infty, \tau_b^- = \infty\}} \right)}. \quad (6.12)$$

Using Theorem 3.1, we can compute the numerator as

$$n^U \left(e^{-q\tau_a^+} : \tau_a^+ < \infty \right) = n^X \left(e^{-q\tau_a^+} : \tau_a^+ < \infty \right) = \frac{1}{W_X^{(q)}(a)}. \quad (6.13)$$

We divide the denominator $n^U \left(1 - e^{-qT_0} 1_{\{\tau_a^+ = \infty, \tau_b^- = \infty\}} \right)$ into the following sum:

$$n^U (1 - e^{-qT_0}) + n^U (e^{-qT_0} : \{\tau_a^+ < \infty\} \cap \{\tau_b^- = \infty\}) + n^U (e^{-qT_0} : \{\tau_b^- < \infty\}). \quad (6.14)$$

Let us compute these expectations. For the second term, we have

$$\begin{aligned} & n^U (e^{-qT_0} : \{\tau_a^+ < \infty\} \cap \{\tau_b^- = \infty\}) \\ &= n^U \left(e^{-q\tau_a^+} \left(e^{-qT_0} 1_{\{\tau_b^- = \infty\}} \right) \circ \theta_{\tau_a^+} : \tau_a^+ < \infty \right) \end{aligned} \quad (6.15)$$

$$= n^X \left(e^{-q\tau_a^+} : \tau_a^+ < \infty \right) \mathbb{E}_a^X \left(e^{-q\tau_0^-} \mathbb{E}_{X(\tau_0^-)}^Y \left(e^{-qT_0} 1_{\{T_0 < \tau_b^-\}} \right) : \tau_0^- < \infty \right) \quad (6.16)$$

$$= \frac{1}{W_X^{(q)}(a)} \int \frac{W_Y^{(q)}(u-b)}{W_Y^{(q)}(-b)} G_X^{(q)}(a, du \, dv), \quad (6.17)$$

where in (6.17) we used Theorem 3.1, Theorem 2.2 and (2.12). For the third term, we have

$$n^U (e^{-qT_0} : \tau_b^- < \infty) = n^U \left(e^{-q\tau_0^-} \mathbb{E}_{U(\tau_0^-)}^U (e^{-qT_0} : \tau_b^- < T_0) : \tau_0^- < \infty \right). \quad (6.18)$$

Using Lemma 3.3, we have

$$(6.18) = \int e^{-\Phi_X(q)v} \mathbb{E}_u^Y(e^{-qT_0} : \tau_b^- < T_0) \tilde{\Pi}_X(du \, dv) \quad (6.19)$$

$$= \int e^{-\Phi_X(q)v} \left(e^{\Phi_Y(q)u} - \frac{W_Y^{(q)}(u-b)}{W_Y^{(q)}(-b)} \right) \tilde{\Pi}_X(du \, dv), \quad (6.20)$$

where in (6.20) we used (2.12) and (2.13). Therefore, using (5.15), we obtain

$$n^U \left(1 - e^{-qT_0} 1_{\{\tau_a^+ = \infty, \tau_b^- = \infty\}} \right) = \frac{1}{W_X^{(q)}(a)} \frac{W_U^{(q)}(a, b)}{W_U^{(q)}(0, b)} \quad (6.21)$$

and we obtain (6.10) for $x = 0$. For all $x < 0$, we have

$$\mathbb{E}_x^U \left(e^{-q\tau_a^+} : \tau_a^+ < \tau_b^- \right) = \mathbb{E}_x^Y \left(e^{-q\tau_0^+} : \tau_0^+ < \tau_b^- \right) \mathbb{E}_0^U \left(e^{-q\tau_a^+} : \tau_a^+ < \tau_b^- \right). \quad (6.22)$$

Using (2.12) and (6.10) for $x = 0$, we have (6.10) for $x < 0$. For all $x > 0$, we have

$$\begin{aligned} & \mathbb{E}_x^U \left(e^{-q\tau_a^+} : \tau_a^+ < \tau_b^- \right) \\ &= \mathbb{E}_x^U \left(e^{-q\tau_a^+} : \tau_a^+ < \tau_0^- \right) + \mathbb{E}_x^U \left(e^{-q\tau_0^-} \left(e^{-q\tau_a^+} 1_{\{\tau_a^+ < \tau_b^-\}} \right) \circ \theta_{\tau_0^-} : \tau_0^- < \tau_a^+ \right) \end{aligned} \quad (6.23)$$

$$= \frac{W_X^{(q)}(x)}{W_X^{(q)}(a)} + \mathbb{E}_x^X \left(e^{-q\tau_0^-} \mathbb{E}_{X(\tau_0^-)}^Y \left(e^{-q\tau_a^+} : \tau_a^+ < \tau_b^- \right) : \tau_0^- < \tau_a^+ \right), \quad (6.24)$$

where in (6.24) we used (2.12). Using (6.10) for $x < 0$ and Lemma 2.3, the second term of (6.24) is equal to

$$\int \frac{W_U^{(q)}(u, b)}{W_U^{(q)}(a, b)} \underline{\Gamma}_X^{(q; 0, a)}(x, v) \tilde{\Pi}_X(du \, dv). \quad (6.25)$$

Thus we obtain (6.10) for $x > 0$. The proof is complete. \square

Corollary 6.3. *For all $x \in (-\infty, a]$ and $q \geq 0$, we have*

$$\mathbb{E}_x^U \left(e^{-q\tau_a^+} \right) = \frac{\overline{W}_U^{(q)}(x)}{\overline{W}_U^{(q)}(a)} \quad (6.26)$$

where

$$\overline{W}_U^{(q)}(x) = \begin{cases} W_X^{(q)}(x) (\Psi_X'(0) \vee 0) \\ \quad + \int (W_X^{(q)}(x) e^{\Phi_Y(0)u} \\ \quad - W_X^{(q)}(x-v) e^{\Phi_Y(q)u}) \tilde{\Pi}_X(du \, dv), & x \in (0, a] \\ e^{\Phi_Y(q)x}, & x \in (-\infty, 0]. \end{cases} \quad (6.27)$$

In particular, $\overline{W}_U^{(q)}(x)$ is a continuous and increasing function of x .

Proof. Using the monotone convergence theorem and Theorem 6.2, we have

$$\mathbb{E}_x^U(e^{-q\tau_a^+}) = \lim_{b \downarrow -\infty} \mathbb{E}_x^U(e^{-q\tau_a^+} : \tau_a^+ < \tau_b^-) = \lim_{b \downarrow -\infty} \frac{W_U^{(q)}(x, b)/W_Y^{(q)}(-b)}{W_U^{(q)}(a, b)/W_Y^{(q)}(-b)}. \quad (6.28)$$

Using the last equality of [6, pp.124], we have

$$\lim_{b \downarrow -\infty} (W_U^{(q)}(x, b)/W_Y^{(q)}(-b)) = \overline{W}_U^{(q)}(x), \quad (6.29)$$

and we have (6.26).

Next, we prove that $\overline{W}_U^{(q)}$ is increasing and continuous. It is obvious that $\overline{W}_U^{(q)}$ is increasing and continuous on $(-\infty, 0]$, since $\overline{W}_U^{(q)}(x) = e^{\Phi_Y(q)x}$. Using the dominated convergence theorem, we have

$$\lim_{\varepsilon \downarrow 0} \overline{W}_U^{(q)}(\varepsilon) = \lim_{\varepsilon \downarrow 0} \frac{1}{\mathbb{E}_0^U(e^{-q\tau_\varepsilon^+})} = \frac{1}{\mathbb{E}_0^U(\lim_{\varepsilon \downarrow 0} e^{-q\tau_\varepsilon^+})} = 1, \quad (6.30)$$

so that we see $\overline{W}_U^{(q)}$ is continuous at 0. Since

$$\overline{W}_U^{(q)}(x) = \frac{1}{\mathbb{E}_0^U(e^{-q\tau_x^+})}, \quad (6.31)$$

it is thus sufficient to prove that $\mathbb{E}_0^U(e^{-q\tau_x^+})$ is decreasing and continuous on $(0, \infty)$. For $0 < x < y$, we have

$$\mathbb{E}_0^U(e^{-q\tau_x^+}) - \mathbb{E}_0^U(e^{-q\tau_y^+}) = \mathbb{E}_0^U(e^{-q\tau_x^+}) \left(1 - \mathbb{E}_x^U(e^{-q\tau_y^+})\right) \geq 0. \quad (6.32)$$

Using (2.12), for $x > 0$, we have

$$\limsup_{\varepsilon \downarrow 0} \left| \mathbb{E}_0^U(e^{-q\tau_{x-\varepsilon}^+}) - \mathbb{E}_0^U(e^{-q\tau_{x+\varepsilon}^+}) \right| = \limsup_{\varepsilon \downarrow 0} \mathbb{E}_0^U(e^{-q\tau_{x-\varepsilon}^+}) \left(1 - \mathbb{E}_{x-\varepsilon}^U(e^{-q\tau_{x+\varepsilon}^+})\right) \quad (6.33)$$

$$\leq \limsup_{\varepsilon \downarrow 0} \left(1 - \mathbb{E}_{x-\varepsilon}^X(e^{-q\tau_{x+\varepsilon}^+} : \tau_{x+\varepsilon}^+ < \tau_0^-)\right) \quad (6.34)$$

$$= \left(1 - \lim_{\varepsilon \downarrow 0} \frac{W_X^{(q)}(x - \varepsilon)}{W_X^{(q)}(x + \varepsilon)}\right) = 0. \quad (6.35)$$

The proof is complete. □

Theorem 6.4. *Our generalized refracted Lévy process is a Feller process.*

Proof. Since $R_U^{(q)}$ comes from transition operators, it is sufficient to verify the following conditions:

- (i) For all $q > 0$, $R_U^{(q)}$ is a map from C_0 to C_0 .

(ii) For all $f \in C_0$, $\lim_{q \uparrow \infty} \|qR_U^{(q)}f - f\| = 0$.

(iii) For all $q > 0$, $\|qR_U^{(q)}1\| \leq 1$.

(iv) For all $\alpha, \beta > 0$,

$$R_U^{(\alpha)} - R_U^{(\beta)} = (\beta - \alpha)R_U^{(\alpha)}R_U^{(\beta)}. \quad (6.36)$$

Note that (iii) and (iv) are obvious.

1) The proof of (i)

First, we prove that $R_U^{(q)}f$ is continuous. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Noting that U has no positive jump, we have

$$\begin{aligned} & \left| R_U^{(q)}f(x + \varepsilon) - R_U^{(q)}f(x) \right| \\ & \leq \left| R_U^{(q)}f(x + \varepsilon) - \mathbb{E}_x^U \left(e^{-q\tau_{x+\varepsilon}^+} \right) R_U^{(q)}f(x + \varepsilon) \right| + \left| \mathbb{E}_x^U \left(\int_0^{\tau_{x+\varepsilon}^+} e^{-qt} f(U_t) dt \right) \right| \end{aligned} \quad (6.37)$$

$$\leq \left| R_U^{(q)}f(x + \varepsilon) \right| \left(1 - \mathbb{E}_x^U \left(e^{-q\tau_{x+\varepsilon}^+} \right) \right) + \|f\| \mathbb{E}_x^U \left(\int_0^{\tau_{x+\varepsilon}^+} e^{-qt} dt \right) \quad (6.38)$$

$$\leq \frac{2}{q} \|f\| \left(1 - \mathbb{E}_x^U \left(e^{-q\tau_{x+\varepsilon}^+} \right) \right). \quad (6.39)$$

By Corollary 6.3, we have

$$(6.39) = \frac{2}{q} \|f\| \left(1 - \frac{\overline{W}_U^{(q)}(x)}{\overline{W}_U^{(q)}(x + \varepsilon)} \right) \rightarrow 0 \text{ as } \varepsilon \downarrow 0. \quad (6.40)$$

The left-continuity is similar.

Second, we prove that $R_U^{(q)}f$ vanishes at $-\infty$. For $x < 0$, we may rewrite (5.7) as

$$R_U^{(q)}f(x) = R_Y^{(q)}f(x) - e^{\Phi_Y(q)x} R_Y^{(q)}f(0) + e^{\Phi_Y(q)x} R_U^{(q)}f(0). \quad (6.41)$$

By the Feller property of Y , we see that $\lim_{x \downarrow -\infty} R_U^{(q)}f(x) = 0$.

Third, we prove that $R_U^{(q)}f$ vanishes at $+\infty$. We may assume without loss of generality that $f \geq 0$. For all $x > 0$, we have

$$R_U^{(q)}f(x) = \mathbb{E}_x^U \left(\left(\int_0^{\tau_0^-} + \int_{\tau_0^-}^{\infty} \right) e^{-qt} f(U_t) dt \right) \quad (6.42)$$

$$\leq R_X^{(q)}f(x) + \mathbb{E}_x^X \left(e^{-q\tau_0^-} R_U^{(q)}f(X_{\tau_0^-}) \right) \cdot \frac{1}{q} \|f\|. \quad (6.43)$$

By the Feller property of X and by the fact that $\mathbb{E}_x^X \left(e^{-q\tau_0^-} \right) = \mathbb{E}_0^X \left(e^{-q\tau_x^-} \right) \rightarrow 0$ as $x \rightarrow \infty$, we obtain $\lim_{x \uparrow \infty} R_U^{(q)}f(x) = 0$.

2) The proof of (ii)

Define

$$\omega_\varepsilon(f; x) = \sup_{y: |y-x| \leq \varepsilon} |f(y) - f(x)|. \quad (6.44)$$

First, we prove the pointwise convergence:

$$\lim_{q \uparrow \infty} qR_U^{(q)} f(x) = f(x), \quad x \in \mathbb{R}. \quad (6.45)$$

For $x \in \mathbb{R}$ and $\varepsilon > 0$, we have

$$\left| qR_U^{(q)} f(x) - f(x) \right| \quad (6.46)$$

$$\leq q\mathbb{E}_x^U \left(\left(\int_0^{\tau_{x+\varepsilon}^+ \wedge \tau_{x-\varepsilon}^-} + \int_{\tau_{x+\varepsilon}^+ \wedge \tau_{x-\varepsilon}^-}^\infty \right) e^{-qt} |f(U_t) - f(x)| dt \right) \quad (6.47)$$

$$\leq \mathbb{E}_x^U \left(1 - e^{-q(\tau_{x+\varepsilon}^+ \wedge \tau_{x-\varepsilon}^-)} \right) \omega_\varepsilon(f; x) + 2 \|f\| \mathbb{E}_x^U \left(e^{-q(\tau_{x+\varepsilon}^+ \wedge \tau_{x-\varepsilon}^-)} \right). \quad (6.48)$$

We thus obtain $\limsup_{q \uparrow \infty} \left| qR_U^{(q)} f(x) - f(x) \right| \leq \omega_\varepsilon(f; x)$ for all $\varepsilon > 0$, which proves (6.45). By a standard argument with the help of the fact that the dual space of C_0 can be identified with the space of signed measures, we can see that $R_U^{(p)}(C_0)$ is dense in C_0 for all $p > 0$.

Let $f = R_U^{(1)} g$ for some $g \in C_0$. Using the resolvent equation, we have

$$\left\| f - qR_U^{(q)} f \right\| = \left\| R_U^{(q)} g - R_U^{(q)} f \right\| \leq \frac{1}{q} \|g - f\| \rightarrow 0, \text{ as } q \uparrow \infty. \quad (6.49)$$

Since $R_U^{(1)}(C_0)$ is dense in C_0 , we obtain claim (ii).

The proof is now complete. \square

7 Potential measure of killed refracted Lévy processes

In this section, we calculate the potential measure of refracted Lévy processes killed on exiting $[b, a]$.

Theorem 7.1. *For all $x \in [b, a]$ and $q \geq 0$, we have*

$$\bar{\mathbb{E}}_U^{(q)}(x, y) = \begin{cases} \frac{W_U^{(q)}(x, b)}{W_U^{(q)}(a, b)} W_X^{(q)}(a - y) - W_X^{(q)}(x - y), & y \in (0, a] \\ \frac{W_U^{(q)}(x, b)}{W_U^{(q)}(a, b)} W_U^{(q)}(a, y) - W_U^{(q)}(x, y), & y \in [b, 0). \end{cases} \quad (7.1)$$

Proof. We follow the notation of Lemma 6.1 for L , η , κ , etc.

Step.1 We calculate in the case $x = 0$. When X has unbounded variation paths, we have

$$\begin{aligned} & \mathbb{E}_0^U \left(\int_0^{\tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt \right) \\ &= \mathbb{E}_0^U \left(\int_{(0, \infty)} e^{-qs} 1_{\{s < \eta(\kappa_{A \cup B})\}} dL(s) \right) n^U \left(\int_0^{T_0 \wedge \tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt \right). \end{aligned} \quad (7.2)$$

where we used the compensation theorem of the excursion point process. We may rewrite (7.2) using η' , as

$$\mathbb{E}_0^U \left(\int_0^\infty e^{-q\eta'(t)} 1_{\{t < \kappa_{A \cup B}\}} dt \right) n^U \left(\int_0^{T_0 \wedge \tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt \right), \quad (7.3)$$

the first factor of which equals to

$$\int_0^\infty e^{-tn^U(1 - e^{-qT_0:(A \cup B)^c})} e^{-tn^U(A \cup B)} dt = \frac{1}{n^U(1 - e^{-qT_0} 1_{\{(A \cup B)^c\}})}. \quad (7.4)$$

When X has bounded variation paths, we have

$$\begin{aligned} & \mathbb{E}_0^U \left(\int_0^{\tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt \right) \\ &= \sum_{n=0}^\infty \mathbb{E}_0^U \left(\int_{T_0^{(n)}}^{T_0^{(n+1)} \wedge \tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt : T_0^{(n)} < \tau_a^+ \wedge \tau_b^- \right) \end{aligned} \quad (7.5)$$

$$= \sum_{n=0}^\infty \mathbb{E}_0^U (e^{-qT_0} : \tau_a^+ = \infty, \tau_b^- = \infty)^n \mathbb{E}_0^U \left(\int_0^{T_0 \wedge \tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt \right) \quad (7.6)$$

$$= \frac{\mathbb{E}_0^U \left(\int_0^{T_0 \wedge \tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt \right)}{1 - \mathbb{E}_0^U (e^{-qT_0} : \tau_a^+ = \infty, \tau_b^- = \infty)}, \quad (7.7)$$

where we used the notation of the proof of Theorem 5.1. Since $n^U = \delta_X \mathbb{E}_0^U$, we obtain

$$\mathbb{E}_0^U \left(\int_0^{\tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt \right) = \frac{n^U \left(\int_0^{T_0 \wedge \tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt \right)}{n^U (1 - e^{-qT_0} : \tau_a^+ < \infty, \tau_b^- = \infty)}, \quad (7.8)$$

which has the same form as in the case of unbounded variation. The denominator has already computed in (6.21). Let us compute the numerator. In the case $f = 1_{(a', a]}$ for $0 < a' < a$, we have

$$n^U \left(\int_0^{T_0 \wedge \tau_a^+ \wedge \tau_b^-} e^{-qt} 1_{\{U_t \in (a', a]\}} dt \right) = n^X \left(e^{-q\tau_{a'}^+} : \tau_{a'}^+ < \infty \right) \mathbb{E}_{a'}^X \left(\int_0^{\tau_a^+ \wedge \tau_0^-} e^{-qt} 1_{\{X_t \in (a', a]\}} dt \right) \quad (7.9)$$

$$= \frac{1}{W_X^{(q)}(a)} \int_{(a', a]} W_X^{(q)}(a - y) dy \quad (7.10)$$

where in (7.10) we used Theorem 3.1 and (2.18). Thus we obtain (7.1) for $x = 0$ and $y \in (0, a]$. In the case $f = 1_{[b, b']}$ for $b < b' < 0$, we have

$$n^U \left(\int_0^{T_0 \wedge \tau_a^+ \wedge \tau_b^-} e^{-qt} 1_{\{U_t \in [b, b']\}} dt \right) = n^X \left(e^{-q\tau_0^-} \mathbb{E}_{X(\tau_0^-)}^Y \left(\int_0^{\tau_b^- \wedge T_0} e^{-qt} 1_{\{Y_t \in [b, b']\}} dt \right) : \tau_0^- < \tau_a^+ \right). \quad (7.11)$$

Using Lemma 3.4, we have

$$(7.11) = \int_b^{b'} \left(\int \bar{\mathbb{E}}_Y^{(q; b, 0)}(u, y) \frac{W_X^{(q)}(a - v)}{W_X^{(q)}(a)} \tilde{\Pi}_X(du \, dv) \right) dy. \quad (7.12)$$

Using (6.21), (7.10) and (7.12), we obtain (7.1) for $x = 0$ and $y \in [b, 0)$.

Step.2 We calculate in the case $x < 0$. We have

$$\begin{aligned} & \mathbb{E}_x^U \left(\int_0^{\tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt \right) \\ &= \mathbb{E}_x^Y \left(\int_0^{\tau_0^+ \wedge \tau_b^-} e^{-qt} f(Y_t) dt \right) + \mathbb{E}_x^U \left(e^{-q\tau_0^+} \left(\int_0^{\tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt \right) \circ \theta_{\tau_0^+} : \tau_0^+ < \tau_b^- \right). \end{aligned} \quad (7.13)$$

Using (2.15), we have that the first term equals to

$$\int_b^0 f(y) \bar{\mathbb{E}}_Y^{(q; b, 0)}(x, y) dy. \quad (7.14)$$

Using (2.12) and **Step.1**, we have that the second term equals to

$$\mathbb{E}_x^Y \left(e^{-q\tau_0^+} : \tau_0^+ < \tau_b^- \right) \mathbb{E}_0^U \left(\int_0^{\tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt \right) \quad (7.15)$$

$$= \frac{W_U^{(q)}(x, b)}{W_U^{(q)}(a, b)} \left(\int_0^a f(y) W_X^{(q)}(a - y) dy \right) \quad (7.16)$$

$$+ \int_b^0 f(y) \left(\int W_X^{(q)}(a - v) \times \bar{\mathbb{E}}_Y^{(q; b, 0)}(u, y) \tilde{\Pi}_X(du \, dv) \right) dy. \quad (7.17)$$

Using (7.14) and (7.17), we obtain (7.1) for $x < 0$.

Step.3 We calculate in the case $x > 0$. We have

$$\begin{aligned} & \mathbb{E}_x^U \left(\int_0^{\tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt \right) \\ &= \mathbb{E}_x^X \left(\int_0^{\tau_0^- \wedge \tau_a^+} e^{-qt} f(X_t) dt \right) + \mathbb{E}_x^U \left(e^{-q\tau_0^-} \left(\int_0^{\tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt \right) \circ \theta_{\tau_0^-} : \tau_0^- < \tau_a^+ \right). \end{aligned} \quad (7.18)$$

Using (2.18), we have the first term equals to

$$\int_0^a f(y) \bar{\Gamma}_X^{(q;0,a)}(x, y) dy. \quad (7.19)$$

The second term equals to

$$\begin{aligned} & \mathbb{E}_x^X \left(e^{-q\tau_0^-} \mathbb{E}_{X(\tau_0^-)}^U \left(\int_0^{\tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt \right) : \tau_0^- < \tau_a^+ \right) \\ &= \int \mathbb{E}_u^U \left(\int_0^{\tau_a^+ \wedge \tau_b^-} e^{-qt} f(U_t) dt \right) \bar{\Gamma}_X^{(q;0,a)}(x, v) \tilde{\Pi}_X(du dv), \end{aligned} \quad (7.20)$$

where in (7.20) we used Lemma 2.3. If f is 0 on $(-\infty, 0]$, we have

$$(7.20) = \left(\frac{W_U^{(q)}(x, b)}{W_U^{(q)}(a, b)} - \frac{W_X^{(q)}(x)}{W_X^{(q)}(a)} \right) \int_0^\infty f(y) W_X^{(q)}(a - y) dy. \quad (7.21)$$

From (7.19) and (7.21) we obtain (7.1) for $x > 0$ and $y \in (0, a]$. If f is 0 on $(0, \infty)$, we have

$$(7.20) = \int_0^\infty f(y) \left(\frac{W_U^{(q)}(x, b)}{W_U^{(q)}(a, b)} W_U^{(q)}(a, y) - W_U^{(q)}(x, y) \right) dy. \quad (7.22)$$

Thus we obtain (7.1) for $x > 0$ and $y \in [b, 0)$. \square

8 Approximation problem

Let Z be a spectrally negative Lévy process. Let Ψ_Z denote the Laplace exponent represented by (2.10). For $n \in \mathbb{N}$, we define

$$\begin{aligned} \Psi_{Z^{(n)}}(q) &= \gamma_Z q - \sigma_Z^2 n^2 \left(1 - e^{q(-\frac{1}{n})} + q \left(-\frac{1}{n} \right) \right) \\ &\quad - \int_{(-\infty, -\frac{1}{n})} \left(1 - e^{qy} + qy 1_{(-1, -\frac{1}{n})}(y) \right) \Pi_Z(dy) \end{aligned} \quad (8.1)$$

$$= \delta_{Z^{(n)}} q - \int_{(-\infty, 0)} (1 - e^{qy}) \Pi_{Z^{(n)}}(dy) \quad (8.2)$$

where

$$\delta_{Z^{(n)}} = \gamma_Z + \sigma_Z^2 n + \int_{(-1, -\frac{1}{n})} (-y) \Pi_Z(dy) \quad (8.3)$$

$$\Pi_{Z^{(n)}} = 1_{(-\infty, -\frac{1}{n})} \Pi_Z + \sigma_Z^2 n^2 \delta_{(-\frac{1}{n})}. \quad (8.4)$$

If we denote by $Z^{(n)}$ a Lévy process with Laplace exponent $\Psi_{Z^{(n)}}$, it is actually a compound Poisson process with positive drift. We note that $\Psi_{Z^{(n)}}(q) \rightarrow \Psi_Z(q)$ for all $q \geq 0$, so that

we have $Z^{(n)} \rightarrow Z$ in law on \mathbb{D} . More precisely, by Bertoin [2, pp.210], we see that there exists a coupling of $Z^{(n)}$'s such that $Z^{(n)} \rightarrow Z$ uniformly on compact intervals almost surely, which we will call the *uniformly convergent coupling*.

Let X and Y be spectrally negative Lévy processes, in addition, X has unbounded variation paths and no Gaussian component. For each $n \in \mathbb{N}$, let $X^{(n)}$ and $Y^{(n)}$ be independent Lévy processes with Laplace exponents $\Psi_{X^{(n)}}$ and $\Psi_{Y^{(n)}}$, respectively. Let $U^{(n)}$ be defined as a unique strong solution of the stochastic differential equation

$$U_t^{(n)} = U_0^{(n)} + \int_{(0,t]} 1_{\{U_{s-}^{(n)} \geq 0\}} dX_s^{(n)} + \int_{(0,t]} 1_{\{U_{s-}^{(n)} < 0\}} dY_s^{(n)}. \quad (8.5)$$

Theorem 8.1. $\{(U^{(n)}, \mathbb{P}_x^{U^{(n)}})\}_{n \in \mathbb{N}}$ converges in distribution to U under \mathbb{P}_x^U for all $x \in \mathbb{R}$.

Remark 8.2. We see

$$\delta_{X^{(n)}} \mathbb{P}^{X^{(n)}} \rightarrow n^X \text{ and } \delta_{X^{(n)}} \mathbb{P}^{U^{(n)}} \rightarrow n^U. \quad (8.6)$$

The precise statements are as follows: For all bounded continuous function f , we have

$$\delta_{X^{(n)}} \mathbb{E}_0^{X^{(n)}} \left(\int_0^{T_0} e^{-qt} f(X_t^{(n)}) dt \right) \rightarrow n^X \left(\int_0^{T_0} e^{-qt} f(X_t) dt \right) \text{ as } n \uparrow \infty \quad (8.7)$$

and

$$\delta_{X^{(n)}} \mathbb{E}_0^{U^{(n)}} \left(\int_0^{T_0} e^{-qt} f(U_t^{(n)}) dt \right) \rightarrow n^U \left(\int_0^{T_0} e^{-qt} f(U_t) dt \right) \text{ as } n \uparrow \infty. \quad (8.8)$$

Proofs of these formulas are easy, so we omit it.

Lemma 8.3. For all non-positive $x^{(n)}$ and x satisfying $x^{(n)} \rightarrow x$ as $n \uparrow \infty$ and for all $q > 0$ and bounded continuous function f , we have

$$R_{Y^{(n)}0}^{(q)} f(x^{(n)}) \rightarrow R_{Y0}^{(q)} f(x) \text{ as } n \uparrow \infty. \quad (8.9)$$

Proof. Using strong Markov property, we have

$$R_{Y^{(n)}0}^{(q)} f(x^{(n)}) = R_{Y^{(n)}}^{(q)} f(x^{(n)}) - \mathbb{E}_{x^{(n)}}^{Y^{(n)}} (e^{-q\tau_0^+}) R_{Y^{(n)}0}^{(q)} f(0) \quad (8.10)$$

and a similar identity for $R_{Y0}^{(q)} f(x)$. Using the uniformly convergent coupling and the dominated convergence theorem, we have $R_{Y^{(n)}}^{(q)} f(x^{(n)}) = R_{x^{(n)}+Y^{(n)}}^{(q)} f(0) \rightarrow R_{x+Y}^{(q)} f(0) = R_Y^{(q)} f(x)$. Since $\Psi_{Y^{(n)}} \rightarrow \Psi_Y$ pointwise as $n \uparrow \infty$, we have $\Phi_{Y^{(n)}} \rightarrow \Phi_Y$ pointwise as $n \uparrow \infty$ and thus

$$\mathbb{E}_{x^{(n)}}^{Y^{(n)}} (e^{-q\tau_0^+}) = e^{-\Phi_{Y^{(n)}}(q)x^{(n)}} \rightarrow e^{-\Phi_Y(q)x} = \mathbb{E}_x^Y (e^{-q\tau_0^+}). \quad (8.11)$$

Thus we obtain (8.9). \square

Theorem 8.4. For all $x \in \mathbb{R}$, $q > 0$ and bounded continuous function f , we have

$$R_{U^{(n)}}^{(q)} f(x) \rightarrow R_U^{(q)} f(x) \text{ as } n \uparrow \infty. \quad (8.12)$$

Proof. We may assume without loss of generality that $0 \leq f \leq 1$. We write $\rho_X := \inf_{n \in \mathbb{N}} \Phi_{X^{(n)}}(q)$ and $\rho_Y := \inf_{n \in \mathbb{N}} \Phi_{Y^{(n)}}(q)$. Since $\Phi_Z(q)$ is strictly positive for all spectrally negative Lévy process Z , we have ρ_X and ρ_Y are strictly positive.

We prove (8.12) for $x = 0$. By (5.6) and (5.5) of Theorem 5.1, it is sufficient to prove

$$\int_0^\infty e^{-\Phi_{X^{(n)}}(q)y} f(y) dy \rightarrow \int_0^\infty e^{-\Phi_X(q)y} f(y) dy \quad (8.13)$$

and

$$\int R_{Y^{(n)0}}^{(q)} f(u) K_{X^{(n)}}^{(q)}(du \, dv) \rightarrow \int R_{Y^0}^{(q)} f(u) K_X^{(q)}(du \, dv). \quad (8.14)$$

Using $\Phi_{X^{(n)}} \rightarrow \Phi_X$ and the dominated convergence theorem, we have (8.13). Let us prove (8.14). Using (3.8) with $c = 1$ and changing variables, we have

$$\begin{aligned} & \int R_{Y^{(n)0}}^{(q)} f(u) K_{X^{(n)}}^{(q)}(du \, dv) \\ &= \int_{(-\infty, 0)} \Pi_X(du) 1_{(u < -\frac{1}{n})} \int_0^{-u} e^{-\Phi_{X^{(n)}}(q)v} R_{Y^{(n)0}}^{(q)} f(u+v) dv \end{aligned} \quad (8.15)$$

and a similar identity for (Y^0, X) . We have

$$\begin{aligned} & \left| 1_{(u < -\frac{1}{n})} \int_0^{-u} e^{-\Phi_{X^{(n)}}(q)v} R_{Y^{(n)0}}^{(q)} f(u+v) dv \right| \\ & \leq \int_0^{-u} e^{-\rho_X v} \mathbb{E}_{u+v}^{Y^{(n)}} \left(\int_0^{T_0} e^{-qt} dt \right) dv \end{aligned} \quad (8.16)$$

$$\leq \frac{1}{q} \int_0^{-u} e^{-\rho_X v} \left(1 - e^{\Phi_{Y^{(n)}}(q)(u+v)} \right) dv \quad (8.17)$$

$$\leq \frac{1}{q} \int_0^{-u} e^{-\rho_X v} \left(1 - e^{\rho_Y(u+v)} \right) dv \quad (8.18)$$

$$\leq \frac{1}{q\rho_X} (1 - e^{\rho_X u}) (1 - e^{\rho_Y u}) \in L^1(\Pi_X). \quad (8.19)$$

Thus we may apply the dominated convergence theorem to obtain

$$\lim_{n \uparrow \infty} (8.15) = \int_{(-\infty, 0)} \Pi_X(du) \int_0^{-u} e^{-\Phi_X(q)v} \left(R_{Y^0}^{(q)} f(u+v) \right) dv, \quad (8.20)$$

which shows (8.14).

For $x < 0$, (8.12) is obvious by (5.7) of Theorem 5.1 and Lemma 8.3.

We prove (8.12) for $x > 0$. By (5.8) of Theorem 5.1, it suffices to prove

$$\underline{R}_{X^{(n)}}^{(q;0)} f(x) \rightarrow \underline{R}_X^{(q;0)} f(x) \quad \text{as } n \uparrow \infty \quad (8.21)$$

and

$$\mathbb{E}_x^{X^{(n)}} \left(e^{-q\tau_0^-} R_{U^{(n)}}^{(q)} f \left(X_{\tau_0^-}^{(n)} \right) \right) \rightarrow \mathbb{E}_x^X \left(e^{-q\tau_0^-} R_U^{(q)} f \left(X_{\tau_0^-} \right) \right) \quad \text{as } n \uparrow \infty. \quad (8.22)$$

For the sample paths of the uniformly convergent coupling, we have $\tau_0^- = \tau_0^-(X^{(n)})$ is constant for large n , so that we have (8.21) by the dominated convergence theorem. By the strong Markov property, we have

$$\begin{aligned} \mathbb{E}_x^{X^{(n)}} \left(e^{-q\tau_0^-} R_{U^{(n)}}^{(q)} f \left(X_{\tau_0^-}^{(n)} \right) \right) \\ = \mathbb{E}_x^{X^{(n)}} \left(e^{-q\tau_0^-} R_{Y_0^{(n)}}^{(q)} f \left(X_{\tau_0^-}^{(n)} \right) \right) + \mathbb{E}_x^{X^{(n)}} \left(e^{\Phi_{Y^{(n)}}(q)X^{(n)}(\tau_0^-) - q\tau_0^-} \right) R_{U^{(n)}}^{(q)} f(0). \end{aligned} \quad (8.23)$$

For the first term we have

$$\lim_{n \uparrow \infty} \mathbb{E}_x^{X^{(n)}} \left(e^{-q\tau_0^-} R_{Y_0^{(n)}}^{(q)} f \left(X_{\tau_0^-}^{(n)} \right) \right) = \mathbb{E}_x^X \left(e^{-q\tau_0^-} R_{Y_0}^{(q)} f \left(X_{\tau_0^-} \right) \right) \quad (8.24)$$

where we used the dominated convergence theorem and Lemma 8.3. \square

Let C_0 denote the set of continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which vanish at $+\infty$ and $-\infty$. Note that C_0 is a Banach space with respect to the supremum norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$ for $f \in C_0$. For all stochastic process Z , $t > 0$, $x \in \mathbb{R}$ and positive or bounded measurable function f , define

$$P_t^Z f(x) := \mathbb{E}_x^Z(f(Z_t)). \quad (8.25)$$

Theorem 8.5. *For all $q > 0$, $t > 0$ and $f \in C_0$, we have*

$$R_{U^{(n)}}^{(q)} f \rightarrow R_U^{(q)} f \quad \text{uniformly as } n \uparrow \infty, \quad (8.26)$$

$$P_t^{U^{(n)}} f \rightarrow P_t^U f \quad \text{uniformly as } n \uparrow \infty. \quad (8.27)$$

Proof of Theorems 8.1 and 8.5. After we obtain (8.26), using Corollary 8.5, Theorem 6.4 and [11, Theorem 3.4.2], we have (8.27). Using [5, Theorem 19.25], it is true that $\{(U^{(n)}, \mathbb{P}_x^{U^{(n)}})\}_{n \in \mathbb{N}}$ converges in distribution to U under \mathbb{P}_x^U for all $x \in \mathbb{R}$.

Let us prove (8.26). We divide the proof of (8.26) into several steps.

Step.1 Let $k > 0$ be a constant. We prove $\{\overline{W}_{U^{(n)}}^{(q)}(x)\}_{n \in \mathbb{N}}$ is equicontinuous uniformly in $x \in [-k, k]$. For this, we prove a pointwise convergence $\lim_{n \uparrow \infty} \overline{W}_{U^{(n)}}^{(q)}(x) = \overline{W}_U^{(q)}(x)$. Since $\{\overline{W}_{U^{(n)}}^{(q)}\}_{n \in \mathbb{N}}$ is increasing and continuous by Corollary 6.3, the pointwise convergence implies convergence uniformly in $x \in [-k, k]$, thus $\{\overline{W}_{U^{(n)}}^{(q)}(x)\}_{n \in \mathbb{N}}$ is equicontinuous uniformly in $x \in [-k, k]$. The desired convergence is obvious for $x \leq 0$ by the definition of $\overline{W}_U^{(q)}(x)$.

For $x \leq 0$, it suffices to show

$$\lim_{n \uparrow \infty} \mathbb{E}_0^{U(n)} \left(e^{-q\tau_x^+} \right) = \mathbb{E}_0^U \left(e^{-q\tau_x^+} \right) \quad (8.28)$$

by Corollary 6.3. By the strong Markov property, we have

$$R_{U(n)}^{(q)} 1_{(-\infty, x)}(0) = \frac{1}{q} \left(1 - \mathbb{E}_0^{U(n)} \left(e^{-q\tau_x^+} \right) \right) + \mathbb{E}_0^{U(n)} \left(e^{-q\tau_x^+} \right) R_{U(n)}^{(q)} 1_{(-\infty, x)}(x). \quad (8.29)$$

As $f^- := 1_{(-\infty, x)}$ is not continuous, we take bounded continuous functions such that f_m^- and f_m^+ such that $f_m^- \uparrow f^-$ and $f_m^+ \downarrow f^+ := 1_{(-\infty, x]}$. Using Theorem 8.4, we have $R_{U(n)}^{(q)} f_m^\pm \rightarrow R_U^{(q)} f_m^\pm$. It is obvious that $R_{U(n)}^{(q)} f^\pm \rightarrow R_U^{(q)} f^\pm$. Thus we obtain (8.28).

Step.2 We may assume without loss of generality that $\|f\| = 1$. Let us prove

$$R_{U(n)}^{(q)} f(x) \rightarrow R_U^{(q)} f(x) \text{ uniformly in } x \in [-k, k]. \quad (8.30)$$

Since we have the pointwise convergence by Theorem 8.4, it is sufficient to prove $\{R_{U(n)}^{(q)} f\}_{n \in \mathbb{N}}$ is equicontinuous. For all $x, y \in \mathbb{R}$ with $x < y$, making a computation similar to **1)** of the proof of Theorem 6.4, we have

$$\left| R_{U(n)}^{(q)} f(y) - R_{U(n)}^{(q)} f(x) \right| \leq \frac{2}{q} \|f\| \left(1 - \frac{\overline{W}_{U(n)}^{(q)}(x)}{\overline{W}_{U(n)}^{(q)}(y)} \right). \quad (8.31)$$

Let $\varepsilon > 0$ be a constant. By **Step.1** and since $\inf_{n \in \mathbb{N}} \overline{W}_{U(n)}^{(q)}(-k) = \inf_{n \in \mathbb{N}} e^{-\Phi_{Y(n)}(q)k} > 0$, we see that there exists $\xi > 0$ such that for all $x, y \in [-k, k]$ with $0 < y - x < \xi$

$$\sup_{n \in \mathbb{N}} \left| \overline{W}_{U(n)}^{(q)}(y) - \overline{W}_{U(n)}^{(q)}(x) \right| \leq \varepsilon \inf_{n \in \mathbb{N}} \overline{W}_{U(n)}^{(q)}(-k). \quad (8.32)$$

Then we have

$$(8.31) \leq \frac{2}{q} \|f\| \frac{\varepsilon \inf_{n \in \mathbb{N}} \overline{W}_{U(n)}^{(q)}(-k)}{\overline{W}_{U(n)}^{(q)}(y)} \leq \frac{2}{q} \|f\| \varepsilon \quad (8.33)$$

where we used the fact that $\overline{W}_{U(n)}^{(q)}$ is increasing. Therefor we conclude that $\{R_{U(n)}^{(q)} f\}_{n \in \mathbb{N}}$ is equicontinuous.

Step.3 We prove that for any $\varepsilon > 0$ there is $k > 0$ such that

$$\sup_{x \in (-\infty, -k) \cup (k, \infty)} \sup_{n \in \mathbb{N}} \left| R_{U(n)}^{(q)} f(x) \right| < \varepsilon. \quad (8.34)$$

For all $x < y < 0$ we have

$$\left| R_{U(n)}^{(q)} f(x) \right| = \left| \mathbb{E}_x^{U(n)} \left(\int_0^{\tau_y^+} e^{-qt} f(U_t^{(n)}) dt \right) + \mathbb{E}_x^{U(n)} \left(e^{-q\tau_y^+} \right) R_{U(n)}^{(q)} f(y) \right| \quad (8.35)$$

$$\leq \frac{1}{q} \sup_{z < y} |f(z)| + \frac{1}{q} \sup_{m \in \mathbb{N}} \mathbb{E}_x^{Y(m)} \left(e^{-q\tau_y^+} \right) \|f\|. \quad (8.36)$$

By the same argument, for all $x > y > 0$, we have

$$\left| R_{U(n)}^{(q)} f(x) \right| \leq \frac{1}{q} \sup_{z > y} |f(z)| + \sup_{m \in \mathbb{N}} \mathbb{E}_x^{X(m)} \left(e^{-q\tau_y^-} \right) \|f\|. \quad (8.37)$$

Since $f \in C_0$, there exists $k_1 > 0$ such that

$$\sup_{|z| > k_1} |f(z)| < \frac{1}{3} q \varepsilon. \quad (8.38)$$

Using the uniformly convergence coupling, we have for $x > y > 0$

$$\lim_{n \uparrow \infty} \mathbb{E}_{-x}^{Y(n)} \left(e^{-q\tau_{-y}^+} \right) = \mathbb{E}_{-x}^Y \left(e^{-q\tau_{-y}^+} \right) \quad \text{and} \quad \lim_{n \uparrow \infty} \mathbb{E}_x^{X(n)} \left(e^{-q\tau_y^-} \right) = \mathbb{E}_x^X \left(e^{-q\tau_y^-} \right) \quad (8.39)$$

and

$$\lim_{x \uparrow \infty} \mathbb{E}_{-x}^Y \left(e^{-q\tau_{-y}^+} \right) = 0 \quad \text{and} \quad \lim_{x \uparrow \infty} \mathbb{E}_x^X \left(e^{-q\tau_y^-} \right) = 0. \quad (8.40)$$

By (8.40), there exists $k_2 > k_1$ such that

$$\mathbb{E}_{-k_2}^Y \left(e^{-q\tau_{-k_1}^+} \right) < \frac{\varepsilon}{3 \|f\|} \quad \text{and} \quad \mathbb{E}_{k_2}^X \left(e^{-q\tau_{k_1}^-} \right) < \frac{\varepsilon}{3 \|f\|} \quad (8.41)$$

By (8.39), there exists $N \in \mathbb{N}$ such that for all $n > N$

$$\left| \mathbb{E}_{-k_2}^{Y(n)} \left(e^{-q\tau_{-k_1}^+} \right) - \mathbb{E}_{-k_2}^Y \left(e^{-q\tau_{-k_1}^+} \right) \right| < \frac{\varepsilon}{3 \|f\|} \quad (8.42)$$

and

$$\left| \mathbb{E}_{k_2}^{X(n)} \left(e^{-q\tau_{k_1}^-} \right) - \mathbb{E}_{k_2}^X \left(e^{-q\tau_{k_1}^-} \right) \right| < \frac{\varepsilon}{3 \|f\|}. \quad (8.43)$$

By (8.40) again, there exists $k_3 > k_2$ such that for all $n \leq N$

$$\mathbb{E}_{-k_3}^{Y(n)} \left(e^{-q\tau_{-k_1}^+} \right) < \frac{\varepsilon}{3 \|f\|} \quad \text{and} \quad \mathbb{E}_{k_3}^{X(n)} \left(e^{-q\tau_{k_1}^-} \right) < \frac{\varepsilon}{3 \|f\|} \quad (8.44)$$

Thus we obtain

$$\sup_{n \in \mathbb{N}} \mathbb{E}_{-k_3}^{Y(n)} \left(e^{-q\tau_{-k_1}^+} \right) < \frac{2\varepsilon}{3 \|f\|} \quad \text{and} \quad \sup_{n \in \mathbb{N}} \mathbb{E}_{k_3}^{X(n)} \left(e^{-q\tau_{k_1}^-} \right) < \frac{2\varepsilon}{3 \|f\|}. \quad (8.45)$$

By (8.36), (8.37), (8.38) and (8.45), we obtain (8.34).

The proof is complete. \square

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